

The exactly solvable two - dimensional stationary Schrödinger operators obtaining by the nonlocal Darboux transformation

A. G. Kudryavtsev

Institute of Applied Mechanics,
Russian Academy of Sciences, Moscow 119991, Russia
kudryavtsev_a_g@mail.ru

Abstract

The Fokker-Planck equation associated with the two - dimensional stationary Schrödinger equation has the conservation law form that yields a pair of potential equations. The special form of Darboux transformation of the potential equations system is considered. As the potential variable is a nonlocal variable for the Schrödinger equation that provides the nonlocal Darboux transformation for the Schrödinger equation. This nonlocal transformation is applied for obtaining of the exactly solvable two - dimensional stationary Schrödinger equations. The examples of exactly solvable two - dimensional stationary Schrödinger operators with smooth potentials decaying at infinity are obtained.

1 Introduction

Consider the two - dimensional stationary Schrödinger equation

$$\frac{\partial^2}{\partial x^2} Y(x, y) + \frac{\partial^2}{\partial y^2} Y(x, y) - u(x, y) Y(x, y) = 0 \quad (1)$$

In the case $u = -E + V(x, y)$ equation (1) describes nonrelativistic quantum system with energy E [1]. In the case $u = \omega^2/c(x, y)^2$ equation (1) describes an acoustic pressure field with temporal frequency ω in inhomogeneous media with sound velocity c [2]. The case of fixed energy E for two - dimensional equation is of interest for the multidimensional inverse scattering theory [3] due to connections with two - dimensional integrable nonlinear

systems [4], [5]. The case of fixed frequency ω is of interest for modelling in acoustic tomography [6].

The useful tool for one - dimensional Schrödinger equation is the Darboux transformation [7]. Straightforward generalizations of Darboux transformation for two - dimensional case were proposed including operators of second order in derivatives [8], [9] but set of exactly solvable two - dimensional models obtained is rather limited. Some examples of exactly solvable two - dimensional stationary Schrödinger operators with smooth rational potentials decaying at infinity were obtained in the papers [10], [11] by application of the Moutard transformation which is a two-dimensional generalization of the Darboux transformation [12]. In the past years progress was made in the symmetry group analysis of differential equations by extending the spaces of symmetries of a given partial differential equations system to include non-local symmetries [13], [14]. In the present paper the nonlocal variable is included in Darboux transformation for investigation of exactly solvable two - dimensional stationary Schrödinger equations.

2 The nonlocal Darboux transformation for the Schrödinger equation

Substituting the following expression into equation (1)

$$Y(x, y) = W(x, y) e^{h(x, y)} \quad (2)$$

we obtain the Fokker-Planck equation

$$W_{xx} + W_{yy} + \frac{\partial}{\partial x} (2 h_x W) + \frac{\partial}{\partial y} (2 h_y W) = 0 \quad (3)$$

if condition

$$u = -h_{xx} - h_{yy} + h_x^2 + h_y^2 \quad (4)$$

holds.

The Fokker-Planck equation (3) has the conservation law form that yields a pair of potential equations

$$W_x + 2 h_x W - Q_y = 0 \quad (5)$$

$$W_y + 2 h_y W + Q_x = 0 \quad (6)$$

Let us consider linear operator

$$\hat{L}(h(x, y)) \mathbf{F} = \begin{pmatrix} 2h_x + \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ 2h_y + \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

Consider Darboux transformation in the form

$$\hat{L}_D \mathbf{F} = \begin{pmatrix} r_{11} - a_{11} \frac{\partial}{\partial x} - b_{11} \frac{\partial}{\partial y} & r_{12} - a_{12} \frac{\partial}{\partial x} - b_{12} \frac{\partial}{\partial y} \\ r_{21} - a_{21} \frac{\partial}{\partial x} - b_{21} \frac{\partial}{\partial y} & r_{22} - a_{22} \frac{\partial}{\partial x} - b_{22} \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

If linear operators \hat{L} and \hat{L}_D hold the relation

$$\left(\hat{L}(h(x, y) + \delta h(x, y)) \hat{L}_D - \hat{L}_D \hat{L}(h(x, y)) \right) \mathbf{F} = 0 \quad (7)$$

for any $\mathbf{F} \in \mathcal{F} \supset Ker(\hat{L}(h))$ where $Ker(\hat{L}(h)) = \{\mathbf{F} : \hat{L}(h) \mathbf{F} = 0\}$, then for any $\mathbf{F}_s \in Ker(\hat{L}(h))$ the function $\tilde{\mathbf{F}}(t, x) = \hat{L}_D \mathbf{F}_s(t, x)$ is a solution of the equation $\hat{L}(\tilde{h}) \tilde{\mathbf{F}} = 0$ with new potential $\tilde{h} = h + \delta h$.

If one consider equation (7) on the set \mathcal{F} of arbitrary functions then treating F_1, F_2 and each of its derivatives as independent variables the following equation can be obtained

$$\left(\frac{\partial(h + \delta h)}{\partial x} \right)^2 + \left(\frac{\partial(h + \delta h)}{\partial y} \right)^2 = \left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial y} \right)^2 \quad (8)$$

This is strong limitation for new potential $\tilde{h} = h + \delta h$.

Let us consider equation (7) on the following set of functions: $\mathcal{F}_0 = \{\mathbf{F} : F_{1x} + 2h_x F_1 - F_{2y} = 0\}$. Taking into account this dependance of F_1, F_2 derivatives the equations for $\delta h, r_{ij}, a_{ij}, b_{ij}$ can be obtained. The particular solution of this equations has the form

$$\delta h = -\ln(B(x, y)) \quad (9)$$

$$\hat{L}_D = \begin{pmatrix} BR_1 - B \frac{\partial}{\partial y} & BR_2 \\ B_x - BR_2 & B_y + BR_1 - B \frac{\partial}{\partial y} \end{pmatrix} \quad (10)$$

where R_1, R_2 are expressions in terms of B, h and B satisfy the system of two nonlinear differential equations.

Let us restrict further consideration by the simple case $h = 0$. In this case the expressions for R_1, R_2 have the form

$$R_1 = \frac{B_y (B_{xx} - B_{yy}) - 2 B_x B_{xy}}{2 (B_x^2 + B_y^2)}, \quad R_2 = \frac{B_x (B_{xx} - B_{yy}) + 2 B_y B_{xy}}{2 (B_x^2 + B_y^2)} \quad (11)$$

The system of equations for B have the form

$$\begin{aligned} & - (2 B B_y B_{xy} + B B_x (B_{xx} - B_{yy}) + B_x (B_y^2 + B_x^2)) (B_{xx} + B_{yy}) \\ & \quad + B (B_y^2 + B_x^2) \frac{\partial}{\partial x} (B_{xx} + B_{yy}) = 0 \\ & - (2 B B_x B_{xy} - B B_y (B_{xx} - B_{yy}) + B_y (B_y^2 + B_x^2)) (B_{xx} + B_{yy}) \\ & \quad + B (B_y^2 + B_x^2) \frac{\partial}{\partial y} (B_{xx} + B_{yy}) = 0 \end{aligned} \quad (12)$$

The initial potential u of the Schrödinger equation corresponds to $h = 0$ and according to equation (4) $u = 0$. The new potential of Schrödinger equation corresponds to δh and according to equations (4), (9) is given by

$$\tilde{u}(x, y) = \frac{(B_{xx} + B_{yy})}{B} \quad (13)$$

Note that according to formula (13) B is an example of solution for the Schrödinger equation with potential \tilde{u} . For $h = 0$ initial system of potential equations (5), (6) has the form

$$W_x - Q_y = 0, \quad W_y + Q_x = 0 \quad (14)$$

and according (10) we have for the solution of the new Fokker-Planck equation with potential δh

$$\tilde{W}(x, y) = B R_1 W - B \frac{\partial W}{\partial y} + B R_2 Q \quad (15)$$

where W, Q are solutions of the system of equations (14).

The equations (2), (9) provide for the solution of the new Schrödinger equation with potential \tilde{u} the relation $\tilde{Y} = \tilde{W}/B$. The equation (2) in the case $h = 0$ provide for the initial solution Y of the Schrödinger equation with potential $u = 0$ that $Y = W$. Then according (15) the Darboux transformation for the Schrödinger equation is

$$\tilde{Y}(x, y) = R_1 Y - \frac{\partial Y}{\partial y} + R_2 Q \quad (16)$$

This is nonlocal Darboux transformation as the potential variable Q is a nonlocal variable connected with Y by the system

$$Y_x - Q_y = 0, \quad Y_y + Q_x = 0 \quad (17)$$

To obtain the first example for the solution of equations (12) let us consider ansatz $B = f(xy)$. This ansatz provide in particular the solution

$$B_s = \tanh\left(\frac{xy - C_2}{C_1}\right) \quad (18)$$

where C_1, C_2 are arbitrary constants. Note that B_s and $1/B_s$ are not solutions of the Laplace equation. The solution B_s provides by the formula (13)

$$\tilde{u} = -2 C_1^{-2} (x^2 + y^2) \left(\cosh\left(\frac{xy - C_2}{C_1}\right) \right)^{-2} \quad (19)$$

3 Decaying at infinity smooth rational potentials of Schrödinger equation

The system of equations (12) for B has some properties that help to get its solutions. It can be proved by straightforward calculation that if B is a solution of equations (12) then $1/B$ and CB where C is an arbitrary constant are solutions as well. It is obvious from the form of equations (12) that any solution of the Laplace equation $B_{xx} + B_{yy} = 0$ is the solution of these equations.

The solution B_L of the Laplace equation provide $\tilde{u} = 0$ according to the formula (13). Taking $B = 1/B_L$ one obtains nontrivial \tilde{u} but potentials of this kind have singularities. To avoid singularities the special ansatz for B can be used. Let us consider

$$S_n = \sum_{i=0}^n \frac{p_i (x - x_i) + q_i (y - y_i)}{(x - x_i)^2 + (y - y_i)^2} = \frac{N_n}{M_n} \quad (20)$$

where p_i, q_i, x_i, y_i are arbitrary constants, N_n is the numerator of S_n and the denominator M_n has the form

$$M_n = \prod_{i=0}^n ((x - x_i)^2 + (y - y_i)^2) \quad (21)$$

The function S_n is a solution of the Laplace equation. Let us consider the following ansatz for B

$$B_n = \frac{N_n}{M_n + C} \quad (22)$$

where C is a constant. If $C > 0$ then denominator of B_n is positive.

Substituting the ansatz (22) into equations (12) it can be verified by straightforward calculation that B_n is a solution of equations (12) for small n if N_n is a solution of the Laplace equation.

The first example of the rational solution is

$$B_0 = \frac{p_0 (x - x_0) + q_0 (y - y_0)}{(x - x_0)^2 + (y - y_0)^2 + C} \quad (23)$$

The solution B_0 provides by the formula (13)

$$\tilde{u}_0 = \frac{-8C}{((x - x_0)^2 + (y - y_0)^2 + C)^2} \quad (24)$$

The second example of the rational solution B_1 can be obtained from the formula (22) where

$$\begin{aligned} N_1 = & (p_0 (x_0 - x_1) - q_0 (y_0 - y_1)) (x^2 - y^2) + 2 (p_0 (y_0 - y_1) + q_0 (x_0 - x_1)) xy \\ & - (p_0 (x_0^2 - x_1^2 + y_0^2 - y_1^2) + 2 q_0 (x_0 y_1 - y_0 x_1)) x \\ & + (2 p_0 (x_0 y_1 - y_0 x_1) - q_0 (x_0^2 - x_1^2 + y_0^2 - y_1^2)) y \\ & - p_0 (x_0 (x_1^2 + y_1^2) - x_1 (x_0^2 + y_0^2)) - q_0 (y_0 (x_1^2 + y_1^2) - y_1 (x_0^2 + y_0^2)) \end{aligned} \quad (25)$$

The solution B_1 provides by the formula (13) the potential

$$\tilde{u}_1 = \frac{-32 C ((x - 1/2 (x_0 + x_1))^2 + (y - 1/2 (y_0 + y_1))^2)}{(((x - x_0)^2 + (y - y_0)^2) ((x - x_1)^2 + (y - y_1)^2) + C)^2} \quad (26)$$

In the case $x_0 = 0$, $y_0 = 0$, $x_1 = -8/17$, $y_1 = -2/17$, $C = 160/17$ the potential \tilde{u}_1 coincides with the first example of the Schrödinger equation potential obtained in the papers [10], [11] by twofold application of the Moutard transformation.

Two arbitrary constants p_0, q_0 in the formula (25) yield two-parameter family of solutions B for the Schrödinger equation with potential \tilde{u}_1 . Other examples of solutions for the Schrödinger equation with potential \tilde{u}_1 can be obtained by the formula (16).

For $n = 2$ the numerator N_2 for the solution B_2 of the equations (12) can be obtained from the formula (20) and the condition that N_2 is a solution of the Laplace equation. As for $n = 1$ this yields two-parameter family of solutions B_2 . To avoid lengthy formulae we omit the expression for B_2 and give the result for the potential in the case $x_0 = 0$, $y_0 = 0$

$$\begin{aligned} \tilde{u}_2 &= \frac{-8 C G(x, y)}{((x^2 + y^2) ((x - x_1)^2 + (y - y_1)^2) ((x - x_2)^2 + (y - y_2)^2) + C)^2} \\ G(x, y) &= ((3x - 2k_1)^2 + (3y - 2k_2)^2) (x^2 + y^2) + 6(k_3 - k_4) (x^2 - y^2) \\ &\quad + 12(k_5 + k_6) xy - 4(k_1 k_3 + k_5 y_2 + k_6 y_1) x \\ &\quad - 4(k_2 k_4 + k_5 x_1 + k_6 x_2) y + (x_1^2 + y_1^2) (x_2^2 + y_2^2) \end{aligned} \quad (27)$$

where $k_1 = x_1 + x_2$, $k_2 = y_1 + y_2$, $k_3 = x_1 x_2$, $k_4 = y_1 y_2$, $k_5 = x_1 y_2$, $k_6 = y_1 x_2$.

In the case $x_1 = -\frac{1}{80} - \frac{1}{80} \sqrt{788 + \sqrt{1252969}}$, $y_1 = \frac{-159 + \sqrt{788 + \sqrt{1252969}}}{16 \sqrt{788 + \sqrt{1252969}}}$, $x_2 = -\frac{1}{80} + \frac{1}{80} \sqrt{788 + \sqrt{1252969}}$, $y_2 = \frac{159 + \sqrt{788 + \sqrt{1252969}}}{16 \sqrt{788 + \sqrt{1252969}}}$, $C = 50$ the potential \tilde{u}_2 coincides with the second example of the Schrödinger equation potential obtained in the papers [10], [11] by twofold application of the Moutard transformation.

For $n = 3$ consider the simple case $x_0 = x_2 = x_3 = 0$, $y_0 = y_2 = y_3 = 0$. In this case the solution B has the form

$$\begin{aligned}
B_3 &= \frac{H(x, y)}{(x^2 + y^2)^3 ((x - x_1)^2 + (y - y_1)^2) + C} \\
H(x, y) &= (m_1 p_1 + m_2 q_1) \left((x^2 - y^2)^2 - 4x^2 y^2 \right) \\
&\quad + 4(m_1 q_1 - m_2 p_1) xy (x^2 - y^2) + (m_3 q_1 - m_4 p_1) x (x^2 - 3y^2) \\
&\quad + (m_4 q_1 + m_3 p_1) y (y^2 - 3x^2)
\end{aligned} \tag{28}$$

where $m_1 = x_1 (x_1^2 - 3y_1^2)$, $m_2 = y_1 (y_1^2 - 3x_1^2)$, $m_3 = 2x_1 y_1 (x_1^2 + y_1^2)$, $m_4 = x_1^4 - y_1^4$.

The solution B_3 provides by the formula (13) the potential

$$\tilde{u}_3 = \frac{-128 C (x^2 + y^2)^2 ((x - 3/4 x_1)^2 + (y - 3/4 y_1)^2)}{((x^2 + y^2)^3 ((x - x_1)^2 + (y - y_1)^2) + C)^2} \tag{29}$$

Two arbitrary constants p_1, q_1 in the formula (28) yield two-parameter family of solutions B for the Schrödinger equation with potential \tilde{u}_3 . Other examples of solutions for the Schrödinger equation with potential \tilde{u}_3 can be obtained by the formula (16).

4 Results and Discussion

The nonlocal Darboux transformation for the two - dimensional stationary Schrödinger equation is obtained. It is shown that this nonlocal transformation provides a useful tool for obtaining exactly solvable two - dimensional stationary Schrödinger operators. The examples of exactly solvable two - dimensional stationary Schrödinger operators with smooth rational potentials decaying at infinity are obtained. The values of the arbitrary constants of rational potentials are indicated which provide the examples of solvable Schrödinger operators obtained in the papers [10], [11] of I.A. Taimanov and S.P. Tsarev.

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